# THE PHILOSOPHY OF MATHEMATICS

BY ERIN WILDING-MARTIN

SABBATICAL PROJECT
FALL 2008
PARKLAND COLLEGE

TABLE OF CONTENTS

# INTRODUCTION

My sabbatical project focuses on the philosophy of mathematics. This is a fascinating aspect of mathematics that is often overlooked in graduate mathematics coursework. Every mathematician has a view of the nature of mathematics, even if he or she does not articulate it. This philosophy informs our views on what constitutes mathematical truth, what counts as mathematical knowledge, what mathematical work looks like, and how the general body of mathematical content is created. Non-mathematicians are often surprised to find out that mathematics, like other disciplines, can be approached from more than one philosophical framework. People often think of math as a concrete discipline, with definite right and wrong answers. However, some mathematicians and philosophers do not see it this way. Some see math as the only concrete discipline that can be counted on for absolute truth. Others see it as subject to human error, like theories in the sciences.

My goals for this sabbatical were to learn more about the different philosophies of mathematics, and to create a summary and resource list for my colleagues. I began in July 2008, attending the International Congress on Mathematical Education in Monterrey, Mexico. This conference brought together mathematicians and mathematics educators from all over the world. There were many philosophy-related sessions, including a set of discussion groups on the philosophy of mathematics and its role in mathematics education. I was often the only person from the United States in the room, and it was very interesting to hear how these issues are much more common in educational discussions in other countries. Upon my return, I read numerous primary and secondary sources. I began with secondary works about the philosophy of mathematics, to get a general feel for the field of study. Then I read primary material by several major mathematical philosophers. I also met with Dr. Timothy McCarthy, a philosophy of mathematics professor at the University of Illinois at Urbana-Champaign. In November, I attended the Midwest Philosophy of Mathematics Workshop at the University of Notre Dame. There I met many philosophers of mathematics and watched presentations on major contemporary questions in the field. Finally, I compiled this report to summarize some of the major philosophical opinions and to offer a list of resources. I have included an annotated bibliography to guide those who might be interested in further reading on the subject. I also presented a summary of my project to my colleagues at a meeting for faculty in the Mathematics Department.

Learning about different philosophies of mathematics has allowed me to explore my own views on mathematics and its teaching. It has been fascinating to look at mathematics in different ways, and to think about how I might present these different perspectives in my classroom. Each philosophy provides an alternate view of the mathematical world, and I think each represents an important aspect of the discipline. If I can share different ways of looking at mathematics with my students, I can show them that it is not all about memorizing rules and solving equations. This may capture their interest and allow them to see math as a bigger picture—as something worth exploring, thinking about, and discussing.

# WHAT IS PHILOSOPHY OF MATHEMATICS?

The philosophy of mathematics considers what is behind the math that we do. What is mathematics? Is it some cosmic truth we discover, or is it created by humans? Do mathematical objects such as numbers and functions really exist, or are they just symbols we have invented? Two of the great debates in the history of mathematical philosophy center around ontology and epistemology. Where did mathematics come from? How do we know that it is true?

Where did mathematics come from? Is it discovered or created? Ontological questions are concerned with the nature and status of mathematical objects. Some people believe that the numbers, functions, and other mathematical objects that we talk about actually exist, either in the world or "out there" somewhere. There are things about them that we need to learn. Statements like $2 + 2 = 4$ are true whether we know about them or not, and they are waiting to be discovered. This is called *independent truth.* It can be likened to the perennial question, "If a tree falls in the forest with no one around, does it make a sound?" Here, "If there are no humans to study mathematics, does mathematical truth still exist?" Those who believe in independent truth say yes. Others believe that mathematical symbols have been made up to make our calculations possible, and that mathematical truth is a human invention. These debates can get very heated, sometimes taking on an almost religious tone. To some, denying the existence of independent truth is dangerously close to denying the role of God as creator of all.

How do we know that mathematics is true? How do we know that what we think is true really is? Epistemological questions address issues of mathematical justification and knowledge. In the mathematical community, the primary means of justification is proof. A proof begins with a set of axioms, and generates a sequence of statements, through inference, that ends in the given proposition.[1] The debate that arises surrounds the nature of the knowledge that results. The traditional philosophy of mathematics has been one which asserts that mathematics offers absolute certainty through proof.[2] This certainty is called *absolute truth*. But how can we be sure that what we have come to know is true? Some philosophers say we are sure because we proved it using deduction and reasoning. Others say we are sure because of the way mathematics models scientific phenomena. There is also a group of philosophers who believe that we *can't* be sure that what we know is actually true, but that it doesn't matter.

In this report, I will summarize several positions in the philosophy of mathematics, both historical and contemporary. The diversity of opinions provides an intriguing look at the world of mathematics. These philosophers offer a sense that there is more going on behind "$x+5=8$" than meets the eye.

---

[1] Paul Ernest, *Social Constructivism as a Philosophy of Mathematics* (Albany: State University of New York Press, 1998), 6.

[2] Kurt Stemhagen, "Beyond Absolutism and Constructivism: The Case for an Evolutionary Philosophy of Mathematics Education" (PhD diss., University of Virginia, 2004), 25.

## HISTORICAL FIGURES IN MATHEMATICAL PHILOSOPHY

The traditional philosophy of mathematics has been one which asserts that mathematics offers absolute certainty or truth.[3] This view can be traced all the way back to Plato, and still persists today. However, there have been differing opinions about what, exactly, mathematics is and how we access it. Here I will discuss three influential philosophers in the history of mathematics.

*Plato*

Plato believed in a metaphysical view of mathematics in which the mathematician must open his mind to grasp the independent, eternal truth that is waiting to be revealed. Mathematical objects are not created in the mind of the mathematician for their utility, but exist eternally in an intangible world which Plato calls the world of Being. The number 2, the perfect sphere, and so on, all exist in the world of Being. When we use these concepts, we use imperfect representations to *refer* to the real object. For example, numbers themselves are part of the world of Being, but we use physical numbers to count objects. Mathematicians deal with numbers themselves, "those numbers that can be grasped only in thought and can't be dealt with in any other way."[4] Similarly, geometry is not about anything in the physical world. Students of geometry use figures of different shapes, but they are imperfect reflections of the real shapes.[5] Mathematical statements, then, refer to real objects and are either true or false—it is up to the mathematician to discover which.

---

[3] Ibid.
[4] Plato, *Republic*, trans. G. M. A. Grube, in *Plato: Complete Works*, ed. John M. Cooper (Indianapolis: Hackett Publishing, 1997), 526a.
[5] Ibid., 510d.

*Aristotle*

Aristotle, on the other hand, believed that the mathematical realm *is* the physical realm. Aristotle believed in independent, absolute truth, but he believed that mathematical objects exist in the empirical world, in physical objects.[6] Generalizations in geometry can be made by considering only those properties of an object that all such objects have in common.[7] For example, a student draws a circle on a piece of paper. She ignores the irrelevant properties such as the color of the paper and ink, the thickness of the line, and the fact that her "circle" tilts to one side. Instead, she considers what would be true if it were a perfect circle: that it would measure 360 degrees around, that its radius would be half its diameter, etc. Then she can use it as an illustration to study the properties of circles in general. Like Plato, Aristotle sees mathematical truth as independent of the knower, waiting to be discovered. But unlike Plato, he finds this truth through the study of perceptible objects.

The influence of Plato and Aristotle has continued throughout history. In the seventeenth century, two major schools of philosophy were prominent. The rationalists, heirs of Plato, found knowledge through reason. The empiricists, influenced by Aristotle, found knowledge through experience. Both schools agreed that mathematical objects exist in the physical world, and that mathematical truth exists independently of the knower. They differed, however, as to how they thought knowledge about mathematical objects was accessed.[8] Rationalists believed that mathematics takes place in the rational mind. Empiricists studied only the perceived objects themselves, believing knowledge is gained through sensory experience.[9]

[6] Stewart Shapiro, *Thinking About Mathematics: The Philosophy of Mathematics* (New York: Oxford University Press, 2000), 64.
[7] Ibid., 68.
[8] Ibid., 75.
[9] Ibid.

*Kant*

In the eighteenth century, new developments in mathematics such as physics, algebraic geometry, and calculus presented new challenges for mathematical philosophy.[10] Physical applicability and notions of the infinite challenged philosophies of abstract mathematics. Immanuel Kant attempted to synthesize rationalism and empiricism in response to the challenges posed by new mathematical advances. For Kant, intuition is the link between the rational and the empirical. Rational analysis does not provide us with information about the *existence* of the concepts in question.[11] It will allow me to say that *if* a triangle has two congruent sides, it will also have two congruent angles. But it will not tell me if any such triangles actually exist. Intuition provides information about how we perceive space and time.[12] It gives the rational mind information about which triangles can actually exist, so that reason can determine which judgments about them are true. Intuition provides information about what *could* be experienced, allowing us to construct new concepts without sensory experience. For Kant, "mathematical knowledge is the knowledge gained by reason from the *construction* of concepts."[13] Mathematical knowledge is gained through the construction of particular examples, combined with the use of reason to abstract and prove universal truths from those constructions. Kant's view has provided the seeds for modern constructivist theories of mathematical learning, which hold that students must mentally construct mathematical concepts for themselves rather than passively absorbing explanations from the teacher.

The work of Plato, Aristotle, and Kant still influences mathematical philosophy today. Twentieth and twenty-first century philosophies tend to reflect the assumption that mathematics

[10] Ibid., 73.
[11] Ibid., 79.
[12] Ibid., 81.
[13] Immanuel Kant, *Critique of Pure Reason*, trans. Norman Kemp Smith (Boston: Macmillan, 1965), B741. Italics in the original.

is certain, a source of absolute truth. However, there are also philosophers who are turning away from this, holding instead that mathematics is created by humans and is therefore fallible. Next, I will discuss the major positions in contemporary mathematical philosophy.

## MODERN ABSOLUTIST PHILOSOPHIES

In the twentieth century and today, there are three philosophies of mathematics that tend to dominate discussion: logicism, formalism, and intuitionism. All are traditional in the sense that they find absolute truth in mathematics. However, they offer differing views on the origins of mathematical content.

*Logicism*

Logicism is the view that all mathematical concepts can be defined and all theorems can be derived using the concepts and axioms of logic, through definition and deduction.[14] Important logicists have included Gottlob Frege and Bertrand Russell. While differing on many points, they agreed that arithmetic can be expressed using logic. For example, the number two is defined as

$$2(P) \equiv (\exists x)(\exists y)\ [P(x) \cdot P(y) \cdot x \neq y \cdot (z)(P(z) \supset. z = x \vee z = y)]$$

> (read: 'there are two *P*s if and only if there are *x*, *y* such that *x* is *P* and *y* is *P* and *x* is not the same thing as *y* and for all *z*, if *z* is *P* then *z* is the same as *x* or *z* is the same as y'.)[15]

Logical positivism is a form of logicism which incorporates empiricism. The logical positivists, such as Rudolf Carnap and A. J. Ayer, view mathematics as a language. Mathematical truth is found through knowledge of the rules of mathematical language. Thus truth is objective and absolute within these rules, but has no factual content about the world.[16] Factual knowledge must be gained through experience. While logicism's cumbersome and restrictive nature has caused it to fall out of favor as a comprehensive philosophy of mathematics, its influence is still seen in the field of mathematical logic.

---

[14] Rudolf Carnap, "The Logicist Foundations of Mathematics," in *Philosophy of Mathematics: Selected Readings*, ed. Paul Benacerraf and Hilary Putnam, 2nd ed. (Cambridge: Cambridge University Press, 1983), 41.
[15] Paul Benacerraf and Hilary Putnam, "Introduction," in *Philosophy of Mathematics: Selected Readings*, ed. Paul Benacerraf and Hilary Putnam, 2nd ed. (Cambridge: Cambridge University Press, 1983), 15.
[16] Shapiro, 129-31.

*Formalism*

Formalism, associated with the work of David Hilbert, is the philosophy that mathematics is nothing but the manipulation of meaningless symbols within a finite axiomatic system. Mathematical truth is determined by the rules of the system in which one is working—to generate truth, such systems must be consistent and complete. Consistency insures that it is not possible to derive logically contradictory statements from the given axioms, and completeness guarantees that all logical truths can be proved within the system.[17] Formalists believe that mathematics is a purely academic enterprise, like deriving an internally consistent system of physics based on the assumption that the world is flat—intellectually stimulating, but not referring to anything in the real world. Mathematics is done purely for the sake of mathematics.

*Intuitionism*

Intuitionism is a form of constructivism, which holds that "mathematics is a production of the human mind."[18] Intuitionism strives to avoid any metaphysical basis for mathematics. There is no independent reality that contains mathematical objects and truths. Without humans, mathematics would not exist—it is constructed by us, not discovered by us. We mentally construct the system of mathematics, and make logical inferences to construct mathematical statements. Despite the view that there is no independent mathematical truth, intuitionists still believe there is absolute truth if mathematics is constructed properly. They are more restrictive in their methods of proof, holding that one cannot prove that the existence of something is mathematically possible without describing how to construct it. For example, I cannot prove that a particular kind of number exists just by showing that it *has* to exist because something else is true. I must actually provide an example, or describe how an example could be created. A major

[17] Richard Wiebe, "Gödel's Theorem (Part II)," *The Two-Year College Mathematics Journal* 6, no. 3 (1975): 4.
[18] Arend Heyting, "The Intuitionist Foundation of Mathematics," in *Philosophy of Mathematics: Selected Readings*, ed. Paul Benacerraf and Hilary Putnam, 2nd ed. (Cambridge: Cambridge University Press, 1983), 52.

tenet of this philosophy is that the law of the excluded middle cannot be assumed, and therefore neither can the resulting principle of double negation. L. E. J. Brouwer explains that one cannot assume that a statement and its negation are mutually exclusive just on general principle, without knowing to which objects the statement refers.[19] Intuitionists endeavor to rebuild mathematics, using what they see as "safe" methods of proof.

While these positions outline the major traditional views in the philosophy of mathematics, there are also a growing number of philosophers who stray from the idea that mathematical truth is absolute. Instead, they believe that mathematics is fallible. These philosophers have differing views on whether independent truth exists, but they share the view that our mathematical knowledge is never certain.

[19] Shapiro, 179.

## MODERN FALLIBILIST PHILOSOPHIES

While the traditional view is that mathematical truth is absolute, some mathematical philosophers disagree. Some of these philosophers believe that independent mathematical truth exists, but we can never be sure if we have accessed it correctly. Others see mathematics as a set of human conventions which do not refer to an independent set of truths. In both cases, mathematics is seen as fallible rather than certain.

The term "fallibilism" was first applied to the philosophy of mathematics by Imre Lakatos.[20] Borrowing heavily from Karl Popper's account of empirical science, Lakatos developed a philosophy of mathematics called quasi-empiricism. This view holds that any mathematical system depends on a set of assumptions that cannot be proved.[21] Mathematical knowledge results from a process of conjecture, proof, and refutation. As in science, mathematical theories are posed, tested, and refined, and are subject to falsification later, in light of new information.[22] Lakatos calls his philosophy *quasi*-empirical because counterexamples are generated abstractly, not observed in the spatio-temporal world.[23] The work of Lakatos has contributed much to the development of fallibilist philosophies of mathematics.

Most of the predominant fallibilist philosophies of mathematics can be categorized as some form of constructivism.[24] These views can be traced back to the work of Kant, a lineage they share with intuitionism. What separates fallibilist constructivism from intuitionism, however, is the view that mathematical judgments are not guaranteed any certainty through logical inference. Different areas of constructivism account differently for the origin of

---

[20] Ernest, *Social Constructivism*, 10.
[21] Ibid., 25.
[22] Marilyn Nickson, "The Culture of the Mathematics Classroom: An Unknown Quantity?" in *Cultural Perspectives on the Mathematics Classroom*, ed. Stephen Lerman (Dordrecht, The Netherlands: Kluwer Academic Publishers, 1994), 11.
[23] Reuben Hersh, *What is Mathematics, Really?* (New York: Oxford University Press, 1997), 213.
[24] Stemhagen, 47.

mathematical concepts. Two major categories are represented by Ernst von Glasersfeld's radical constructivism and Paul Ernest's social constructivism.

*Radical Constructivism*

Von Glasersfeld's radical constructivism is based on the Kantian idea that the status of a knowledge claim cannot be determined by comparing it to some external reality. There is no way to make this objective comparison, because we do not have access to any such reality outside of our experience.[25] There might be an external reality, and there might not. We have no way of knowing. Instead, reality and truth are constructed in the individual mind. Von Glasersfeld's account of this construction comes from a psychological standpoint, incorporating his own version of Piaget's theory of cognitive development. To von Glasersfeld, reality is constructed gradually through our experiences. Knowledge organizes our experiential world—it does not *discover* an external reality.[26] Therefore, knowledge cannot be transmitted from one person to another: it is actively constructed in the human mind, either individually or in response to interaction and discussion with others.[27]

Mathematical knowledge, then, is also a human construction for von Glasersfeld. It is not some external, independent truth. Rather it is a system of concepts and symbols that have been given meaning through human construction. It is not possible to construct just any mathematical ideas you want, though, because they are subject to corroboration from others in the mathematical community. For example, 2 + 2 = 4 is a fact that is certain because we have

---

[25] Ernst von Glasersfeld, *Radical Constructivism: A Way of Knowing and Learning* (London: Falmer Press, 1995), 4.

[26] Ernst von Glasersfeld, "An Exposition of Constructivism: Why Some Like it Radical," in *Constructivist Views on the Teaching and Learning of Mathematics*, ed. Robert B. Davis, Carolyn A. Maher, and Nel Noddings, Journal for Research in Mathematics Education Monograph 4 (Reston, VA: National Council of Teachers of Mathematics, 1990), 22-23.

[27] D. C. Phillips, "An Opinionated Account of the Constructivist Landscape," in *Constructivism in Education: Opinions and Second Opinions on Controversial Issues*, ed. D. C. Phillips, pt. 1 of *Ninety-Ninth Yearbook of the National Society for the Study of Education* (Chicago: National Society for the Study of Education, 2000), 7.

come to construct units in a certain way and have agreed on how they are to be counted.[28] We can be sure of facts that logically follow from this construction. For someone's own construction to be considered correct, it would have to be compatible and produce these facts. Therefore, even though mathematics is a human construction, it is still objective and true. However, this is not an absolute, certain truth because it could later be found faulty.

*Social Constructivism*

Ernest's social constructivism is based on the same premise as radical constructivism—that knowledge is constructed, not discovered. Social constructivists, however, believe that all knowledge is determined by social convention.[29] What counts as knowledge is decided by human interaction and social reality. Ernest argues that mathematics is fallible because its foundation is a set of assumptions that cannot be proved, and because its system of deduction is uncertain. Mathematics is built on a set of assumptions, and is deduced from those assumptions using rules of inference that we must assume are sound.[30] Because we are working under agreed-upon assumptions, mathematics is actually determined by social agreement. Thus, whereas von Glasersfeld focuses on the individual construction of personal knowledge, Ernest focuses on the social construction of shared knowledge.

Ernest sees the traditional, absolutist view of mathematics as a futile search for an objective, independent truth that we can never know for sure. Ernest does believe in an external reality, but not in an absolute knowledge of it. Mathematical knowledge is a human creation, and is therefore fallible. In Ernest's social constructivism, the human element is based in mathematical language, which has developed over time as the mathematical community has

---

[28] Ernst von Glasersfeld, "Aspects of Radical Constructivism and Its Educational Recommendations," in *Theories of Mathematical Learning*, ed. Leslie P. Steffe and Pearla Nesher (Mahwah, NJ: Lawrence Erlbaum Associates, 1996), 313.

[29] Phillips, 6.

[30] von Glasersfeld, "Exposition of Constructivism," 28.

come to agree on the meanings of terms, symbols, syntax, and proof, along with rules for their use.[31] Ernest also draws from his interpretation of Lakatos' work to explain how mathematical knowledge is generated. This is a cycle called the Logic of Mathematical Discovery (LMD), in which a conjecture is proved, tested, and is subject to revision if counterexamples emerge.[32] This cycle repeats until the person is satisfied with her proof. Then she presents it to the mathematical community for public critique and the cycle begins again. In this way, mathematics is created as a body of knowledge which has come to be accepted by the mathematical community as rigorously proven.[33]

An important aspect of Ernest's philosophy is the social responsibility of mathematics. He argues that all knowledge is generated by human intellectual activity, and as part of this knowledge, mathematics is culture-bound and value-laden.[34] Mathematics is not a collection of absolute truths, but rather a system that has been created through social exchange. Because it is defined by the mathematical community, cultural values and preferences are involved in its formation.[35] This incorporates a social responsibility, with ethical consequences. One aspect of this is the responsibility for the consequences of mathematical developments and applications. Another aspect is a responsibility for the conventions and values in the discipline. There are equity issues involved in the determination of what counts as valuable mathematics, what counts as rigorous proof, and who counts as a member of the mathematics community.

These fallibilist philosophies of mathematics are interesting, because instead of focusing on math as absolute truth, they reflect the way mathematics as a discipline has evolved over time. Regardless of whether some cosmic mathematical truth exists, it is mathematicians who have put

---

[31] Ernest, *Social Constructivism*, 79.
[32] Ibid., 112.
[33] Ibid., 144.
[34] Paul Ernest, *The Philosophy of Mathematics Education* (London: Falmer Press, 1991), 262-63.
[35] Ernest, *Social Constructivism*, 270.

forth conjectures and, as a community, have determined which of those conjectures have been proved satisfactorily and will be deemed part of the body of accepted mathematics. As humans, they can make mistakes. Sometimes a crucial error is discovered many years later, and a proof is ultimately rejected or revised.[36] Fallibilist philosophies may sound strange at first, but they seem less so if we just think about the body of mathematics that humans know about, and how it has been put together.

[36] Ibid., 29.

# CONCLUSION

This project has provided me with many new ways of thinking about mathematics. This is fascinating to me not only as a mathematician, but also as a mathematics teacher. While there are many elements that impact the way someone teaches mathematics, one of these is philosophy. Teachers have a philosophy of mathematics, even if they have not explicitly thought about it. The way they think about math influences the way they present it to their students. For example, a Platonist may value traditional lecture and an emphasis on mathematical reasoning, while an Aristotelian may prefer examples and hands-on demonstrations. A formalist might tend to focus on mathematical rules and procedures. A fallibilist might believe that students need to experience the creation of mathematical concepts through open-ended problem solving activities. To me, it seems that all of these are important, interesting aspects of mathematics. Mathematical thinking is about reasoning, inferring information from examples, understanding the patterns in rules and procedures, and discovering new ideas while working on a difficult problem. This project has helped me to reflect on these different aspects and how I can help my students to experience mathematics as a multi-dimensional discipline.

# ANNOTATED BIBLIOGRAPHY FOR FURTHER READING

## General Surveys of Philosophy of Mathematics

Benacerraf, Paul, and Hilary Putnam, eds. *Philosophy of Mathematics: Selected Readings*. 2nd ed. Cambridge: Cambridge University Press, 1983.

This is a collection of articles by some of the major figures in mathematical philosophy. It is a somewhat difficult text, requiring both philosophical and mathematical background to understand. However, it offers an excellent overview of some of the major issues and positions in the philosophy of mathematics.

Hersh, Reuben. *What Is Mathematics, Really?* New York: Oxford University Press, 1997.

A very readable survey of the history of mathematical philosophy. Along the way, Hersh argues for the view that mathematics is a social phenomenon that is a part of human culture and history.

Nasar, Sylvia, and David Gruber. "Manifold Destiny: A Legendary Problem and the Battle over Who Solved It." *New Yorker*, August 28, 2006. http://www.newyorker.com/archive/2006/08/28/060828fa_fact2 (accessed September 10, 2008).

An interesting article about a Fields Medal winner and the controversy surrounding his work. It is a modern illustration of disagreements in the mathematics community over what counts as mathematical proof.

Shapiro, Stewart. *Thinking About Mathematics: The Philosophy of Mathematics*. New York: Oxford University Press, 2000.

A nice overview of major positions in the history of mathematical philosophy. Accessible to readers who have a degree in mathematics, though a little philosophical background would be helpful.

**Specific Philosophies of Mathematics**

Aristotle. *Metaphysics*. Internet Classics Archive. http://classics.mit.edu/Aristotle/metaphysics.html (accessed March 28, 2009).

Books XIII and XIV of Aristotle's *Metaphysics* are largely dedicated to a discussion of the nature of mathematical objects and a rejection of the Platonist position.

Ernest, Paul. *Social Constructivism as a Philosophy of Mathematics*. Albany: State University of New York Press, 1998.

Ernest's second book, which focuses predominantly on his philosophy of mathematics. He offers an in-depth description of his social constructivist philosophy, including an analysis of major philosophical influences such as Wittgenstein and Lakatos.

Kant, Immanuel. *Critique of Pure Reason*. Translated by Norman Kemp Smith. Boston: Macmillan, 1965.

This major philosophical work addresses Kant's epistemology and metaphysics, which includes a discussion of mathematics. It requires a philosophical background.

Kitcher, Philip. *The Nature of Mathematical Knowledge*. New York: Oxford University Press, 1983.

Kitcher outlines a naturalistic philosophy of mathematics, claiming that mathematics develops historically and that mathematical knowledge is empirical.

Lakatos, Imre. *Mathematics, Science and Epistemology*. Edited by John Worrall and Gregory Currie. Vol. 2 of *Philosophical Papers*. Cambridge: Cambridge University Press, 1978.

A collection of papers by Lakatos, elaborating on different aspects of his quasi-empiricist philosophy of mathematics. Also includes some critical essays on the philosophy of science, politics, and education.

Lakatos, Imre. *Proofs and Refutations: The Logic of Mathematical Discovery*. Edited by John Worrall and Elie Zahar. 1976. Reprint, Cambridge: Cambridge University Press, 1999.

Lakatos's famous dialogue which uses a fictional classroom to illustrate his philosophy on the development of mathematics.

Plato. *Plato: Complete Works.* Edited by John M. Cooper. Indianapolis: Hackett Publishing, 1997.

Plato's *Republic* contains some of his ideas on the philosophy of mathematics. He discusses what he believes mathematical objects are, and how they are connected to perfect objects, or Forms, in a metaphysical world of Being.

Putnam, Hilary. *Mathematics, Matter and Method.* Vol. 1 of *Philosophical Papers*. Cambridge: Cambridge University Press, 1979.

Contains a collection of Putnam's papers on the philosophy of mathematics and science. These papers are written from a realist perspective, which holds that statements are true or false regardless of our knowledge of that status. Unlike Platonists, however, Putnam does not hold that mathematical objects exist independently of the human mind.

Steiner, Mark. *Mathematical Knowledge*. Ithaca, NY: Cornell University Press, 1975.

Steiner analyzes and ultimately rejects a version of mathematical logicism. He then devotes the rest of the book to an argument for a Platonist philosophy of mathematics.

Tymoczko, Thomas, ed. *New Directions in the Philosophy of Mathematics: An Anthology*. Rev. ed. Princeton: Princeton University Press, 1998.

A collection of essays arguing for a quasi-empiricist philosophy of mathematics that takes into account mathematical practice.

von Glasersfeld, Ernst. *Radical Constructivism: A Way of Knowing and Learning*. London: Falmer Press, 1995.

Von Glasersfeld outlines his radical constructivist philosophy of mathematics and mathematics education.

Wittgenstein, Ludwig. *Remarks on the Foundations of Mathematics*. Edited by G. H. von Wright, R. Rhees, and G. E. M. Anscombe. Translated by G. E. M. Anscombe. Rev. ed. Cambridge, MA: MIT Press, 1983.

Summarizes much of Wittgenstein's work in the philosophy of mathematics. There are many interpretations of his view, which links the philosophy of mathematics to the philosophy of language.

**Philosophy and the Mathematics Classroom**

Burton, Leone. "Whose Culture Includes Mathematics?" In *Cultural Perspectives on the Mathematics Classroom*, edited by Stephen Lerman, 69-83. Dordrecht, The Netherlands: Kluwer Academic Publishers, 1994.

An interesting article that considers how traditional, western philosophies of mathematics may have a negative impact on the multicultural classroom.

Dossey, John A. "The Nature of Mathematics: Its Role and Its Influence." In *Handbook of Research on Mathematics Teaching and Learning*, edited by Douglas A. Grouws, 39-48. New York: Macmillan, 1992.

A brief overview of how some issues in the philosophy of mathematics have impacted the mathematics classroom.

Ernest, Paul. "The Impact of Beliefs on the Teaching of Mathematics." In *Mathematics Teaching: The State of the Art*, edited by Paul Ernest, 249-54. New York: Falmer Press, 1989.

A short article describing how a teacher's philosophy of mathematics, whether conscious or not, can affect classroom practice.

Ernest, Paul. *The Philosophy of Mathematics Education.* London: Falmer Press, 1991.

Ernest's first comprehensive description of his social constructivist philosophy of mathematics and mathematics education. In the first half, he summarizes his philosophy of mathematics. In the second half of the book, he applies this to educational situations.

Ernest, Paul, ed. *Mathematics, Education and Philosophy: An International Perspective.* London: Falmer Press, 1994.

A collection of articles about the nature of mathematics, describing different social philosophies and their implications for educational practice.

Lampert, Magdalene. "When the Problem Is Not the Question and the Solution Is Not the Answer: Mathematical Knowing and Teaching." *American Educational Research Journal* 27, no. 1 (1990): 29-63.

This article describes a study in which Lampert successfully uses a Lakatosian approach to elementary school mathematics education.

Lerman, Stephen. "Alternative Perspectives of the Nature of Mathematics and Their Influence on the Teaching of Mathematics." *British Educational Research Journal* 16, no. 1 (1990): 53-61.

This article connects different current perspectives in the philosophy of mathematics to their influences on mathematics teaching.

von Glasersfeld, Ernst, ed. *Radical Constructivism in Mathematics Education.* Dordrecht, The Netherlands: Kluwer Academic Publishers, 1991.

A collection of articles pertaining to constructivist theory and its application in mathematics education.