The Fibonacci Sequence

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Recommended Citation
http://spark.parkland.edu/ah/9
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Introduction:

Fibonacci sequence is one of the most famous series of numbers in all of the mathematics. The number sequence is named after Leonardo Fibonacci, who first introduced it to western European mathematics in his 1202 book *Liber Abaci*, although it may have been previously known by ancient Indian mathematics. In mathematical terms, the sequence $F_n$ of Fibonacci numbers is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2},$$

With seed values

$$F_0 = 0 \quad \text{and} \quad F_1 = 1.$$  

As we can see, the first two Fibonacci numbers are 0 and 1, and each subsequent number is the sum of the previous two. The sequence looks like this:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...
Some mathematicians omit zero form the Fibonacci sequence, starting it with two 1’s. The first zero is known as the zeroth Fibonacci number, and has no real practical merit.

**Origins:**

The Fibonacci sequence was known in Indian mathematics independently of the West, but scholars differ on the timing of its discovery. First the Fibonacci sequence was noticed by Pingala, who introduced it in his book *Chandaśāstra*. Later, the number sequence is associated with Indian mathematics; Virahanka, Gopala and Hemachadra. According to Parmanand Singh Virahanka was "the first authority who explicitly gave the rule for the formation" of the Fibonacci numbers. In contrast, Rachel Hall only mentions Hemachandra among these authors as having worked with Fibonacci numbers; she claims that around 1150, Hemachandra noticed that the number of possible rhythms followed the Fibonacci sequence.

In west the sequence was first introduced by Leonardo Fibonacci, who relates the exponential growth of the pair of rabbits with Fibonacci number. The problem is as follows:

Let’s say we have two rabbits. These rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits. The new pair of rabbits also mates at the age of one month, and after another month etc. The rabbits never die, and mating pair always produces a new pair. So how many pairs will be at the end of the $n$th month?

Here is a diagram:
As we can see the number of pairs, as time goes up, is the Fibonacci sequence. After 5\textsuperscript{th} month there are 8 pairs, and after the \(n\)\textsuperscript{th} month the number of pairs of rabbits is equal to the number of new pairs (which is the number of pairs in month \(n-2\)) plus the number of pairs alive last month.

This is the \(n\)\textsuperscript{th} Fibonacci number.

\textbf{The mathematics of Fibonacci numbers:}

Besides the definition shown above the Fibonacci numbers can be represented in many other ways. Here are a few examples of its sum identities:

\[ \sum_{i=0}^{n} F_i = F_{n+2} - 1 \quad \text{The sum of the first } n \text{ Fibonacci numbers is equal to the } (n+2)^{th} \text{ number minus 1.} \]

\[ \sum_{i=0}^{n} iF_i = nF_{n+2} - F_{n+3} + 2 \]
The sum of the squares of the first $n$ Fibonacci numbers is the multiple of the $n^{th}$ and $(n+1)^{th}$ Fibonacci numbers.

Furthermore, Fibonacci numbers obey many mathematical formulas and relations:

The Fibonacci numbers obey the negation formula

$$F_{-n} = (-1)^{n+1} F_n.$$ 

the addition formula

$$F_{m+n} = \frac{1}{2} (F_m L_n + L_m F_n),$$

where $L_n$ is a Lucas number, where $L_n = L_{n-1} + L_{n-2}$ with $L_1 = 1$ and $L_2 = 3$, the subtraction formula

$$F_{m-n} = \frac{1}{2} (-1)^n (F_m L_n - L_m F_n).$$

the fundamental identity

$$L_n^2 - 5 F_n^2 = 4 (-1)^n.$$ 

conjugation relation

$$F_n = \frac{1}{5} (L_{n-1} + L_{n+1}).$$

successor relation

$$F_{n+1} = \frac{1}{2} (F_n + L_n).$$
double-angle formula

\[ F_{2n} = F_n \cdot L_n. \]

multiple-angle recurrence

\[ F_{kn} = L_k F_{k(n-1)} - (-1)^k F_{k(n-2)}. \]

multiple-angle formulas

\[
F_{kn} = \begin{cases} 
\frac{1}{2^{k-1}} \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{i} \binom{2i+1}{i} F_{n+1}^{i+1} F_n^{k-1-2i} L_n^{k-1-2i} & \\
F_n \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \binom{k-1-i}{i} (-1)^i F_n^{k-1-2i} + & \\
\left[ L_n \sum_{i=0}^{\lfloor (k-2)/2 \rfloor} \binom{k-1-i}{i} (-1)^i F_n^{k-1-2i} + \right. \\
\sum_{i=0}^{\lfloor (k/2) \rfloor} \binom{k-i}{i} (-1)^i F_n^{k-1-2i} & \\
& \text{for } k \text{ even} \\
& \text{for } k \text{ odd} \\
= & \\
\sum_{i=0}^{k} \binom{k}{i} F_i F_n^i F_{n-1}^{k-i} \\
= & \\
\sum_{i=0}^{k} \binom{k}{i} F_{-i} F_n^i F_{n+1}^{k-i} \\
\end{cases}
\]

For only \( n \geq 1 \) the extension

\[ F_{k, n+1} = \sum_{i=0}^{k} \binom{k}{i} F_{-i} F_n^i F_{n+1}^{k-i} \]

product expansions
\[ F_m F_n = \frac{1}{5} [L_{m+n} - (-1)^n L_{m-n}] \]

and

\[ F_m L_n = F_{m+n} + (-1)^n F_{m-n} . \]

square expansion,

\[ F^2_n = \frac{1}{5} [L_{2n} - 2 (-1)^n] , \]

and power expansion

\[ F^k_n = \frac{1}{2 \cdot 5^{(k/2)}} \sum_{i=0}^{k} \binom{k}{i} (-1)^{(n+1)} \begin{cases} F_{i-2} \cdot \frac{n}{n} & \text{for } k \text{ odd} \\ \frac{L_{i-2} \cdot n}{n} & \text{for } k \text{ even} \end{cases} \]

Here are given the general relations,

\[
\begin{align*}
F_{n+m} &= F_{n-1} F_m + F_n F_{m+1} , \\
F_{(k+1)n} &= F_{n-1} F_{k n} + F_n F_{k n+1} , \\
F_n &= F_{l} F_{n-l} + F_{l-1} F_{n-l} .
\end{align*}
\]

In the case \( l = n - l + 1 \), then \( l = (n + 1)/2 \) and for \( n \) odd,

\[ F_n = F^2_{(n+1)/2} + F^2_{(n-1)/2} . \]

Similarly, for \( n \) even,

\[ F_n = F^2_{n/2+1} - F^2_{n/2-1} . \]
Furthermore, Fibonacci numbers have many properties. Some of them are fascinating. For example, every 3\textsuperscript{rd} Fibonacci number is an even number, because 3\textsuperscript{rd} Fibonacci number is 2. To put it more generally, every $k$-th Fibonacci number is a multiple of $F(k)$. Moreover, every real positive number $n$ can be represented by the sum of Fibonacci numbers, using one number at most once.

**Relation to the Golden Ratio:**

In mathematics and arts, two quantities are in the golden ratio if the ratio between the sum of those quantities and the larger one is the same as the ratio between the larger one and the smaller. The golden ratio is referred to with the letter $\varphi$ and approximated as 

$$\varphi = \frac{1 + \sqrt{5}}{2} \approx 1.6180339887\ldots$$

The best way to understand this is by using line segment:

$$\begin{align*}
\text{A} & \quad \text{B} \\
\hline
\end{align*}$$

If we arrange $A$ and $B$ such that $\frac{B}{A} = \varphi$ then $(A + B)/B = \varphi$ too.

There are many ways in which the Fibonacci sequence is related to the golden ratio. The main relation is that the further you look at the terms of the sequence, the ratio of the two successive values of terms of Fibonacci sequence becomes closer and closer to golden ratio. The table below shows this.

<table>
<thead>
<tr>
<th>First number</th>
<th>Second number</th>
<th>Ratio $= \frac{\text{Second number}}{\text{First number}}$</th>
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As we can see the ratio of the twelfth and eleventh terms is equal to the golden ratio by four decimal places. As we go further, the ration limit of two successive terms approaches the golden ratio $\varphi$:

$$\lim_{n \to \infty} \frac{F(n + 1)}{F(n)} = \varphi,$$

Furthermore, we can describe this graphically by using squares. Let's take a 1 on 1 square, then add another square to it with the same side. This will create a rectangle with sides 2 and 1. Then we add another square with the side equal to 2. Subsequently, we add more squares whose sides are equal to the longest sides of the resulting rectangles. Here is the picture of the first 6 squares.
We see that the side of the next square is equal to the sum of the sides of previous two squares. The sequence of the length of the sides of successive squares is Fibonacci number. Furthermore, if we draw a quarter circle in each square we get a Fibonacci spiral like this:

**Occurrence in Nature:**

The greatest mystery revolving around Fibonacci sequence is that it occurs in many places in Nature. This is the main reason why it is so famous. Fibonacci numbers appear in nature in different ways. A good example would be the curved section of sea shells. Here is the sketch on the x and y axis of Nautilus shell.
We can see that the spiral part crosses at 1 2 5 on the positive axis, and 0 1 3 on the negative axis. If we look more closely, we notice the oscillatory part of the shell crosses at 0 1 1 2 3 5 on the positive axis, which are the first 6 numbers of Fibonacci sequence! Furthermore, as the spiral grows, it approximately becomes a Fibonacci spiral.

Another example is the petals of flowers. The number of petals on a flower that still has all of its petals intact and has not lost any, for many flowers is a Fibonacci number. Although there are species that have very precise number of petals, most of the flowers have petals, whose numbers are very close to those above, with the average being a Fibonacci number. For example, the flowers that have two or four petals are not common, while there are thousands of species that have one, three or five petals. Here is a list of some common flowers:
- 3 petals: lily, iris
- 5 petals: buttercup, wild rose, larkspur, columbine (aquilegia)
- 8 petals: delphiniums
- 13 petals: ragwort, corn marigold, cineraria,
- 21 petals: aster, black-eyed susan, chicory
- 34 petals: plantain, pyrethrum
- 55, 89 petals: michaelmas daisies, the asteraceae family

The last two examples are generalized, as because of the likelihood of under-development and over-development those flowers may end up with a few less or more petals. In addition, Fibonacci numbers can be seen in the patterns of florets in the head of sunflowers. The sunflower displays florets in spirals in clockwise and counter-clockwise directions.

As shown in the picture, there are 21 florets in clockwise direction, and 34 in counter-clockwise direction. The numbers 34 and 55 are 9th and 10th numbers in the Fibonacci sequence.
In addition, Fibonacci numbers occur in our human bodies. Let’s take human hand for example. Every human has two hands, each one of these has five fingers, each finger has three parts which are separated by two knuckles. All of these numbers fit into the sequence.

In the picture the length of each part of the finger is a Fibonacci number. This example may be a coincidence, considering how close each of the lengths to the corresponding Fibonacci number is. However, we can safely assume that most of the human hands are very close to this one.

The more astonishing example would be the human DNA. DNA is a nucleic acid that contains the genetic instructions used in the development and functioning of all known living organisms, with the exception of some viruses. The main role of DNA molecules is the long-term storage of information. Our bodies grow according to this information stored in the DNA of our every cell. Here is the picture of the cut section of human DNA:
As we see, the DNA looks like the chain of bounded small circles. It measures 34 angstroms long by 21 angstroms wide for each full cycle of its double helix spiral. 34 and 21, of course, are numbers in the Fibonacci series.

Of course, these are only some of the many examples of Fibonacci numbers in nature. With the advancement of technology, scientists are able to view different galaxies, and the astonishing thing is that one category of those galaxies, which are called spiral galaxies, resembles Fibonacci spiral. Here is the picture of M51 galaxies, which is also known as The Whirlpool Galaxy:
If we look closely, we can notice two spiral patterns, and each one of them is approximately a Fibonacci spiral!

The Fibonacci sequence retains a mystique, partly because excellent approximations of it turn up in many unexpected places in nature. The Fibonacci numbers appear in many other unexpected contexts in mathematics, and they continue to spark interest in the scientific community.