Parkland College

A with Honors Projects | Honors Program

2012

Heat Equation

Wyatt Sherlock
Parkland College

Recommended Citation

Sherlock, Wyatt, "Heat Equation" (2012). *A with Honors Projects*. 52.
http://spark.parkland.edu/ah/52

Open access to this Article is brought to you by Parkland College's institutional repository, SPARK: Scholarship at Parkland. For more information, please contact spark@parkland.edu.

Heat Equation

Derivation and Analytical Solution

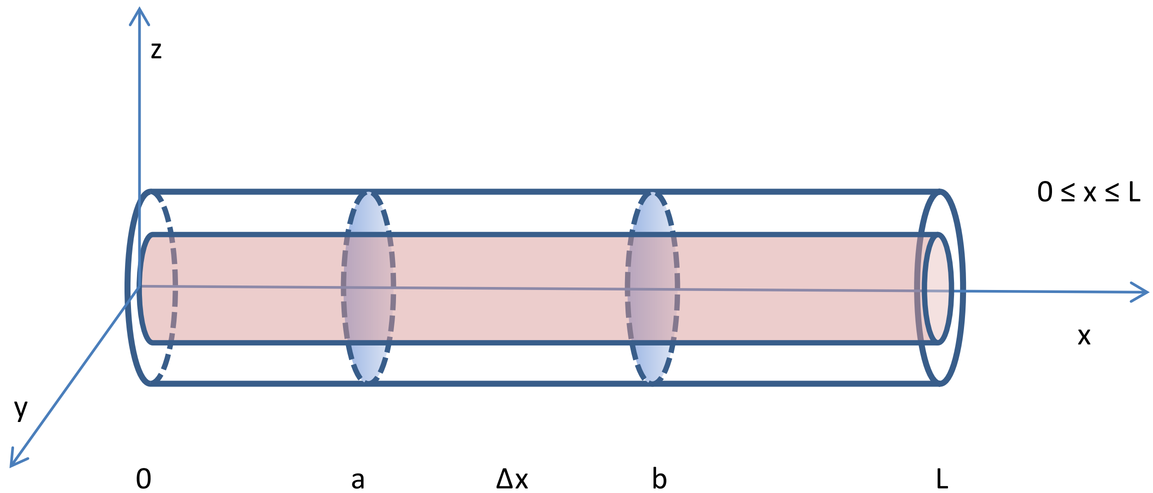
Wyatt Sherlock

Spring 2012

Abstract

This document describes the process of deriving the heat equation from known thermodynamic laws followed by an analytic solution. Towards the end of the document, the heat equation is discussed in terms of practical engineering problems.

Introduction



The cylinder above is a section of thin metal rod with insulation. The inner cylinder is the metal rod while the outer cylinder is rubber insulation. Let's say that the insulation inhibits the flow of heat in the y and z directions so that the heat flows in the x direction only. Since the metal rod is *very* thin, this can be regarded as a one dimensional case. A very thin metal plate would be an example of a two dimensional case.

Deriving the Heat Equation in One Dimension

It is experimentally determined that

$$\Delta Q = c\rho u(x,t)\Delta V$$

Where *Q* is the heat, *c* is the specific heat, ρ is the density, *u* is the temperature as a function of *x* and *t*, and ΔV is a very small volume.

Since the rod is of a uniform diameter, the equation becomes

$$\Delta Q = c\rho u(x,t)A\Delta x$$

where *A* is the cross sectional surface area of the rod. If I integrate both sides from *b* to *a*, the equation becomes

$$\int dQ = \int c\rho A u(x,t)dx$$

$$Q(t) = \int_b^a c\rho A u(x,t)dx$$

where *Q* is a function of time. Differentiating and using Leibniz's Rule

$$\frac{dQ}{dt} = \frac{d}{dt}\int_b^a c\rho A u(x,t)dx = \int_b^a c\rho A \frac{\partial u}{\partial t}dx$$

Fourier's Law states that heat will flow from hot regions to cold regions. Heat is proportional to the negative gradient of the temperature.

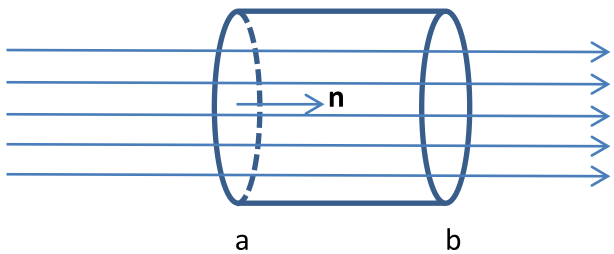
$$\boldsymbol{q} = -kA\boldsymbol{\nabla}u$$

The rate of heat passing through the boundary at position *a* is given by

$$\frac{dQ}{dt} = \int_a kA\boldsymbol{\nabla}u(x,t)\cdot\boldsymbol{n}ds$$

where $\boldsymbol{n}$ is the unit normal vector and *ds* is an infinitesimal surface at *a*. The Divergence Theorem in general is

$$\iint_S \boldsymbol{F}\cdot\boldsymbol{n}ds = \iiint_Q \boldsymbol{\nabla}\cdot\boldsymbol{F}dV$$



$$\boldsymbol{F} = \boldsymbol{\nabla}u(x,t)$$

where *S is* a closed surface from *a* to *b*. *Q* is the small volume from *a* to *b*. Since the vector field (heat) is only flowing in the *x* direction, the theorem can be simplified.

$$\boldsymbol{F} = \boldsymbol{\nabla}u(x,t) = \frac{\partial u}{\partial x}\hat{\imath} + \frac{\partial u}{\partial y}\hat{\jmath} + \frac{\partial u}{\partial z}\hat{k} = \frac{\partial u}{\partial x}\hat{\imath}$$

The flux through the surface at *a* is

$$\int_a \boldsymbol{F}\cdot\boldsymbol{n}ds = \int_b^a \frac{\partial}{\partial x}\hat{\imath}\cdot\frac{\partial u}{\partial x}\hat{\imath}dx$$

$$\frac{dQ}{dt} = \int_a kA\boldsymbol{\nabla}u(x,t)\cdot\boldsymbol{n}ds = \int_b^a kA\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)dx$$

$$\frac{dQ}{dt} = kA\int_b^a \frac{\partial^2 u}{\partial x^2}dx$$

$$\frac{dQ}{dt} = \int_b^a c\rho A\frac{\partial u}{\partial t}dx = kA\int_b^a \frac{\partial^2 u}{\partial x^2}dx$$

$$\int_b^a c\rho A\frac{\partial u}{\partial t}dx = kA\int_b^a \frac{\partial^2 u}{\partial x^2}dx$$

$$\int_b^a c\rho A\frac{\partial u}{\partial t}dx - kA\int_b^a \frac{\partial^2 u}{\partial x^2}dx = 0$$

$$\int_b^a c\rho A\frac{\partial u}{\partial t} - kA\frac{\partial^2 u}{\partial x^2}dx = 0$$

For this to be equal to zero for every choice of *a* and *b* the integrand must be always be identically zero.

$$c\rho A\frac{\partial u}{\partial t} - kA\frac{\partial^2 u}{\partial x^2} = 0$$

The cross sectional areas cancel out

$$c\rho\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

Therefore the heat equation is

$$\frac{\partial u}{\partial t} = \beta\frac{\partial^2 u}{\partial x^2}, \beta = \frac{k}{c\rho}$$

I can show this another way as well. Fourier's Law of heat flow states that heat will diffuse from a hot region to a cold region. Let's say that *a* is held at a higher temperature than *b*. Therefore heat will flow from *a* to *b*. From the perspective of *a*

$$\boldsymbol{q} = -kA\boldsymbol{\nabla}u$$

$$\boldsymbol{q} = -kA\frac{\partial u}{dx}\boldsymbol{x}$$

$$\Delta Q = \frac{-kA[u(a+\Delta x,t) - u(a,t)]}{\Delta x}\Delta t$$

$$\lim_{t\to 0}\frac{\Delta Q}{\Delta t} = \lim_{x\to 0}\frac{-kA[u(a+\Delta x,t) - u(a,t)]}{\Delta x}$$

Since heat is flowing from *a* to *b* the above derivative is positive. The rate of heat flow is

$$\frac{dQ}{dt} = kA\frac{\partial u(a,t)}{\partial x}$$

The rate of heat flow at *a* is proportional to the difference in temperature across the surface at *a*. Now let's look at it from *b*

$$\Delta Q = \frac{-kA[u(b+\Delta x,t) - u(b,t)]}{\Delta x}\Delta t$$

Since the temperature at a *(b+*Δx) is greater than at b,

$$\frac{dQ}{dt} = -kA\frac{\partial u(b,t)}{\partial x}$$

The rate of heat flow at *b* is proportional to the difference in temperature across the surface at *b*. The total rate of heat flowing from *a* to *b* is the input minus the output. This is Fick's Law.

$$\frac{dQ}{dt} = kA\frac{\partial u(a,t)}{\partial x} - kA\frac{\partial u(b,t)}{\partial x}$$

Using the Fundamental Theorem of Calculus

$$\frac{dQ}{dt} = \int_b^a kA \frac{\partial}{\partial x}\frac{\partial u}{\partial x} dx$$

The result is the same

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}, \beta = \frac{k}{c\rho}$$

Solving the Heat Equation

β is called thermal diffusivity and it can vary depending on the type of material used. Let's say that *β* is .05.

$$\beta = .05 \, \frac{\mathrm{m}^2}{s}$$

I am going to set up some initial and boundary conditions for the heat equation. These can vary depending on the problem.

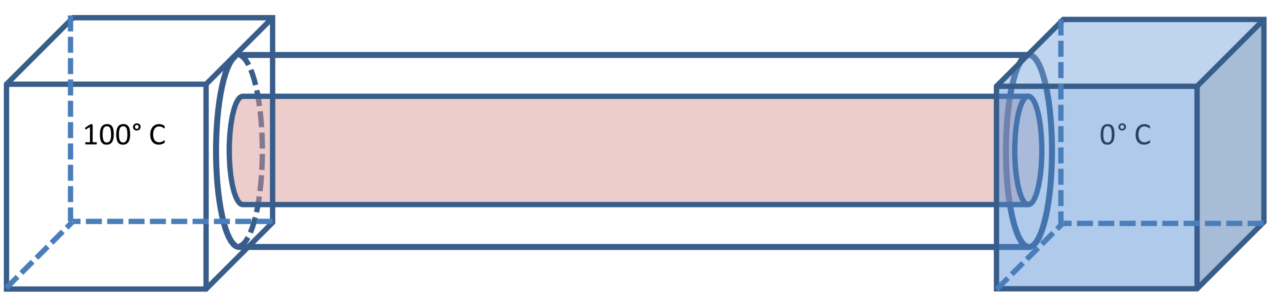
$$u(x,0) = 0 \qquad\qquad 0 < x < L$$

$$u(0,t) = 0$$

$$u(L,t) = 100, L = 4\mathrm{m}$$

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$

So let's say that the left end is submersed in a tank of boiling water kept at 100° C. The right end of the rod is in contact with a block of ice that is always kept at 0° C



The heat equation is linear. I can write the solution as a linear combination of two other solutions. The first solution I can find is the steady state solution. Steady state means that rate of heat flowing into the rod is equal to the rate of heat flowing out of the rod. So the rod is in a state of heat equilibrium. Fourier's Law states that heat will diffuse from hot areas to cold areas. After a long amount of time the rod will be at heat equilibrium, the rate of heat flowing into and out of the rod is the same. As time *t* goes to infinity heat equilibrium will not change. So temperature becomes a function of *x* only. Therefore

$$\frac{\partial u_E}{\partial t} = \frac{\partial^2 u_E}{\partial x^2} = 0, u_E(0) = 0, u_E(4) = 100$$

u_E is the equilibrium temperature. Integrating twice produces

$$u_E = Ax + B, u_E(0) = 0, u_E(4) = 100$$

So plugging in initial conditions I find that *B* is zero and *A* is *25*.

$$u_E(x) = \left(\frac{T_f - T_i}{L}\right)x + T_i$$

$$u_E(x) = \left(\frac{100 - 0}{4}\right)x + 0 = 25x$$

Next I want to look at when heat is initially diffusing through the rod. Let's define the function

$$w(x,t) = u(x,t) - u_E(x)$$

Where *u* is the solution to the heat equation and u_E is the equilibrium condition. Let's take the first and second derivatives with respect to *t*.

$$\frac{\partial w(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial t} - \frac{\partial u_E(x)}{\partial t}$$

$$\frac{\partial^2 w(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$$

So u and w satisfy the same partial differential equation. Now I am going to plug in my initial and boundary conditions.

$$w(x,0) = u(x,0) - 25x = 0 - 25x = -25x$$

$$w(0,t) = u(0,t) - 0 = 0$$

$$w(4,t) = u(4,t) - 100 = 0$$

Now the boundary conditions are homogenous, therefore I can use separation of variables.

$$w(x,t) = X(x)T(t)$$

Plugging back into the heat equation

$$X(x)\frac{dT(t)}{dt} = \beta T(t)\frac{d^2X(x)}{dx^2}$$

Using prime notation

$$\frac{T'}{\beta T} = \frac{X''}{X} = -\Omega$$

For whatever choice of *t* the right hand side must equal to a constant, Ω. The same goes whatever choice of *x*. From the assumption above, these two expressions must equal each other for whatever choice of *x* and *t*. Choosing omega to be negative is just for convenience.

$$T' + \beta\Omega T = 0 \qquad [1]$$

$$X'' + \Omega X = 0 \qquad [2]$$

These are two ordinary differential equations

Solving equation 1

$$\frac{dT}{dt} = -\beta\Omega T$$

$$dT = -\beta\Omega T dt$$

$$\int dT = \int -\beta\Omega T dt$$

$$T(t) = C_1 e^{-\beta\Omega t}$$

Solving equation 2

$$X'' + \Omega X = 0$$

This is a homogeneous second order linear differential equation with constant coefficients. The solution for the differential equation will differ based on the value of Ω.

If $\Omega < 0$

$$\Omega = -\psi^2$$

where ψ is some nonzero constant. I make this substitution because it avoids square roots.

$$X'' - \psi^2 X = 0$$

The characteristic polynomial is

$$r^2 - \psi^2 = 0$$

$$r = \pm\psi$$

$$X(x) = C_2 e^{\psi x} + C_3 e^{-\psi x}$$

Plugging in initial conditions

$$u(0, t) = 0 = u(L, t)$$

$$0 = C_2 + C_3$$

$$C_2 = -C_3$$

$$0 = C_2 e^{\psi L} + C_3 e^{-\psi L}$$

$$0 = C_2(e^{\psi L} - e^{-\psi L})$$

Ω is some constant ≠ 0 so then ψ is ≠ 0. Therefore C_2 must be zero as well as C_3.

$$X(x) = 0$$

This will produce a trivial solution.

Ω = 0 will produce a trivial solution as well.

Now let's say that Ω > 0

$$\Omega = \psi^2$$

$$X'' + \psi^2 X = 0$$

The characteristic polynomial is

$$r^2 + \psi^2 = 0$$

$$r = \pm\psi\mathrm{i}$$

$$X(x) = C_2 cos\psi x + C_3 sin\psi x$$

Plugging in initial conditions

$$u(0,t) = 0 = u(L,t)$$

$$C_2 = 0$$

$$0 = C_3 sin\psi L$$

The sine function is 0 when

$$\psi L = n\pi, n = 0, 1, 2, 3, 4 \dots$$

As a result

$$X(x) = sin\frac{n\pi x}{L} \qquad [1]$$

I can leave off the constant because it will be absorbed into a single constant at the end.

$$\Omega_n = \psi^2 = \left(\frac{n\pi}{L}\right)^2 \qquad [2]$$

Equation 1 is the eigenfunction and equation 2 is the eigenvalue for the problem. Here I left off *n = 0* because it produces zero for any choice of *x*. So

$$n = 1, 2, 3, 4 \dots$$

Hence there are infinitely many solutions to the spatial differential equation.

$$X_n(x) = \sin\frac{n\pi x}{L}$$

The solution is

$$w_n(x,t) = C_n \sin\frac{n\pi x}{L} e^{-\beta\left(\frac{n\pi}{L}\right)^2 t}$$

I have found infinitely many solutions. The solution above satisfies the homogenous boundary conditions. The entire solution is the sum of all the eigenvalues and eigenfunctions. C_n will change for each value of n.

$$w(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} e^{-\beta\left(\frac{n\pi}{L}\right)^2 t}$$

At $t = 0$, the exponential factor equals one.

$$w(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} = -25x$$

$$\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}$$

This is the Fourier sine series. Now I need to find the value of C_n.

First I am going to assume that $w(x)$ is an odd function. This makes sense because sine is also an odd function. I am also going to assume that the series converges to $w(x)$ on the interval $-L \le x \le L$. Next, I want to show that

$$\sin\frac{n\pi x}{L}$$

Is mutually orthogonal for $n = 1,2,3,4\ldots$, on the interval $-L \le x \le L$ and $0 \le x \le L$. Two functions are said to be mutually orthogonal for $m = 1,2,3,4\ldots$

$$\int_a^b f(x)_i g(x)_j dx = \begin{cases} 0, & n \neq m \\ \varepsilon > 0, & n = m \end{cases}$$

Therefore

$$\int_{-L}^{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} dx = 2\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} dx$$

This is true because two odd functions will produce an even function.

Now if $n = m$ then the expression results in

$$2\int_0^L sin\frac{n\pi x}{L}sin\frac{m\pi x}{L}dx = 2\int_0^L sin^2\frac{n\pi x}{L}dx = \int_0^L 1 - cos\frac{2n\pi x}{L}dx = L - sin\frac{2n\pi L}{L} \times \frac{L}{2n\pi}$$

Therefore

$$\int_{-L}^L sin\frac{n\pi x}{L}sin\frac{m\pi x}{L}dx = 2\int_0^L sin\frac{n\pi x}{L}sin\frac{m\pi x}{L}dx = L\,for\,all\,m\,and\,n\,when\,n = m$$

Now if $n \neq m$

$$\int_{-L}^L sin\frac{n\pi x}{L}sin\frac{m\pi x}{L}dx = 2\int_0^L sin\frac{n\pi x}{L}sin\frac{m\pi x}{L}dx$$

Using the trigonometric identity

$$sinAsinB = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$2\int_0^L sin\frac{n\pi x}{L}sin\frac{m\pi x}{L}dx = \int_0^L cos\left(\frac{n\pi x - m\pi x}{L}\right) - cos\left(\frac{n\pi x + m\pi x}{L}\right)dx$$

Therefore

$$\int_0^L cos\left(\frac{n\pi x - m\pi x}{L}\right) - cos\left(\frac{n\pi x + m\pi x}{L}\right)dx = sin\left(\frac{n\pi L - m\pi L}{L}\right) \times \frac{L}{\pi(n - m)} - sin\left(\frac{n\pi L + m\pi L}{L}\right) \times \frac{L}{\pi(n + m)}$$

This becomes

$$sin(\pi(n - m)) \times \frac{L}{\pi(n - m)} - sin\big(\pi(n + m)\big) \times \frac{L}{\pi(n + m)} = 0\,for\,all\,m\,and\,n\,when\,n \neq m$$

$$\int_{-L}^L sin\frac{n\pi x}{L}sin\frac{m\pi x}{L}dx = 2\int_0^L sin\frac{n\pi x}{L}sin\frac{m\pi x}{L}dx = 0$$

Now

$$w(x) = \sum_{n=1}^{\infty} C_n sin\frac{n\pi x}{L} = -25x$$

Multiplying both sides by

$$sin\frac{m\pi x}{L}$$

Produces

$$sin\frac{n\pi x}{L}w(x) = \sum_{n=1}^{\infty} C_n sin\frac{n\pi x}{L}sin\frac{m\pi x}{L}$$

Then integrating

$$\int_{-L}^{L} sin\frac{n\pi x}{L}w(x)dx = \int_{-L}^{L}\sum_{n=1}^{\infty} C_n sin\frac{n\pi x}{L}sin\frac{m\pi x}{L}dx$$

$$\int_{-L}^{L} sin\frac{n\pi x}{L}w(x)dx = \sum_{n=1}^{\infty} C_n \int_{-L}^{L} sin\frac{n\pi x}{L}sin\frac{m\pi x}{L}dx$$

Now the right hand side of the equation is always zero when *n ≠ m*. The equation below is true only when *n = m*. Therefore there will be one non-zero integral.

$$\frac{1}{L}\int_{-L}^{L} sin\frac{m\pi x}{L}w(x)dx = \frac{2}{L}\int_{0}^{L} sin\frac{m\pi x}{L}w(x)dx = C_m, m = 1,2,3,4 \ldots$$

So

$$\frac{2}{L}\int_{0}^{L} sin\frac{n\pi x}{L}w(x)dx = C_n, n = 1,2,3,4 \ldots$$

$$C_n = \frac{2}{4}\int_{0}^{L} sin\frac{n\pi x}{4}(-25x)dx = -\frac{25}{2}\int_{0}^{4} xsin\frac{n\pi x}{4}dx = -\frac{25}{2}\left(\frac{16(\sin(n\pi) - n\pi\cos(n\pi))}{n^2\pi^2}\right)$$
$$= -\frac{25}{2}\left(\frac{-16n\pi(-1)^n}{n^2\pi^2}\right) = \frac{200}{\pi n}(-1)^n$$

Sine is always zero for multiples of pi and cosine alternates between negative one and positive one.

So plugging everything back in, I get

$$w(x) = \sum_{n=1}^{\infty}\frac{200}{\pi n}(-1)^n sin\frac{n\pi x}{L} = -25x$$

$$w(x,t) = \sum_{n=1}^{\infty}\frac{200}{\pi n}(-1)^n sin\frac{n\pi x}{4}e^{-(.05)\left(\frac{n\pi}{4}\right)^2 t}$$

Therefore

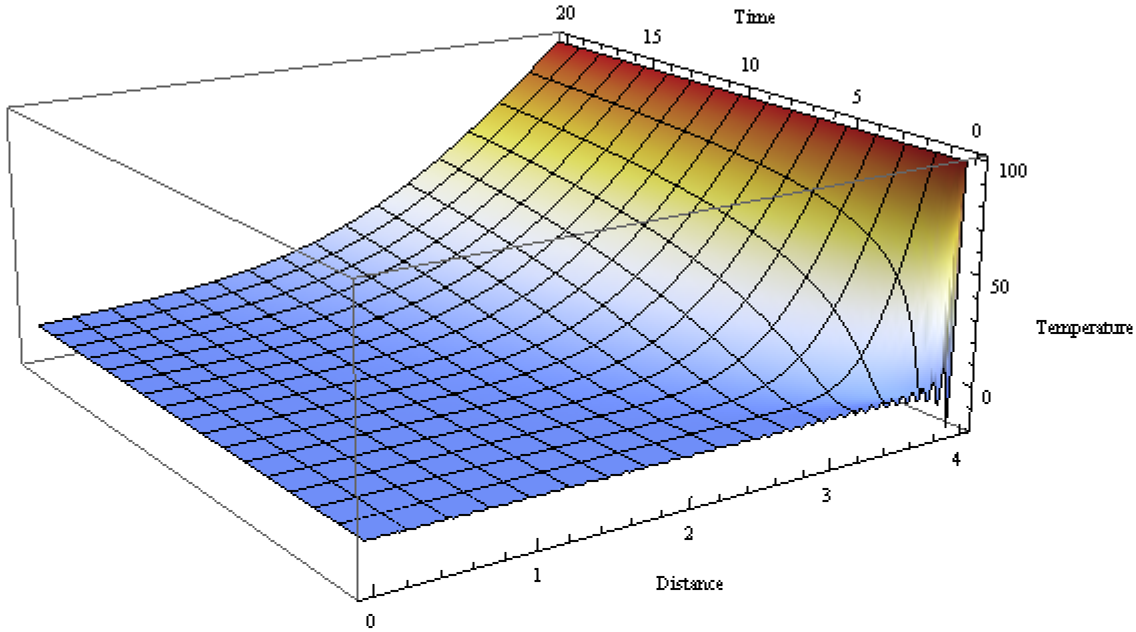
$$u(x,t) = 25x + \sum_{n=1}^{\infty}\frac{200}{\pi n}(-1)^n sin\frac{n\pi x}{4}e^{-(.05)\left(\frac{n\pi}{4}\right)^2 t}$$

As you can see this satisfies the initial condition and the steady state condition

At *t = 0, u(x,t) = 0* because the sum converges to *-25x* for 0 < x < L. As *t* goes to ∞, *u(x,t)* approaches *25x*

It also satifies the boundary conditions at *x = 0, u(x,t) = 0* and at *x = L = 4, u(x,t) = 100.*

The following graph shows the preceeding expression plotted as a function of distance and time. As time goes to infinity, the temperature approaches the steady state condition.



Importance to Engineering

I showed how heat diffuses through a very thin metal rod. This is interpreted as temperature as a function of distance and time. This is not exclusively for thin metal rods, it can be used in two and three dimensions as well. Also boundary and initial conditions can change from example to example.

In the world of engineering, mechanical engineering in particular, this is very important. It allows for the modeling of heat transfer through an engine, for example. Different materials behave differently under high temperature conditions. An engine block machined out of cast iron will respond differently to heat transfer than an aluminum one. Choosing the right type of material is critical in engineering a car. Failure to do so may result in the melting of the engine.

One instance, the Chevrolet Vega had a cast aluminum engine block. This was ideal for fuel consumption and strength however problems with the coolant system ultimately led to engine distortion and failure. MG Midgets had a similar coolant system problem however Midget engine blocks were cast iron and could handle much higher temperatures.

The thermal diffusivity, β, of aluminum and iron:

$$\beta_{Al} = .08418$$

$$\beta_{Fe} = .023$$

Since the thermal diffusivity of aluminum is greater than iron's, an aluminum engine block will heat up much quicker than an iron one. On the following page I have provided two graphs. One shows the temperature as a function of distance and time of a four meter length of aluminum rod. The other is for an iron rod of the same dimension.

The graph on top is of the iron rod and the one on the bottom is of the aluminum one. We can see that at *t = 60 s*, the aluminum rod is very close to the steady state condition while the iron rod is still approaching the steady state condition.