Function of Several Variable

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Function of Several Variables

In 2014 spring semester, I have learnt about the function with one variable so that I want to explore the function with several variables. This essay mainly talks about the function with two variables and their properties.

Definition: f(x,y) Domain: Let D represent domain of f(x,y). The range of F consists of all real numbers f(x,y) Where (x,y) is in D.

Then we solve some problems.

I. Describe the domain of f, and find the indicated function value.

1. \( f(u,v) = \frac{uv}{u-2v} \)

   Domain of f is \( \{(u,v): u \neq 2v\} \)

   \[ f(2,3) = \frac{2 \times 3}{2 - 2 \times 3} = \frac{-3}{2} \]

   \[ f(-1,4) = (-1) \times 4 / (-1) - 2 \times 4 = 4/9 \]

   \[ f(0,1) = 0 \times 1 / 0 = 2 \times 1 = 0 \]

2. \( f(x,y,z) = \sqrt{25 - x^2 - y^2 - z^2} \)

   Domain of f is \( \{(x,y,z): x^2 + y^2 + z^2 \leq 25\} \)

   \[ f(1,-2,2) = \sqrt{25 - 1^2 - (-2)^2 - 2^2} = 4 \]

   \[ f(-3,0,2) = \sqrt{25 - (-3)^2 - 0^2 - 2^2} = 2\sqrt{3} \]
II. Sketch the graph of $f$.

1. $f(x,y) = x^2 + y^2 - 1$

Step: I use an online tool to graph the following graph which is in

www.wolframalpha.com

III. Sketch the level curve of $f$ for the given value of $K$.

1. $f(x,y) = x^2 - y$ $k=0$

Step: we should let $x^2 - y = K = 0$

So we get the $y = x^2$

Then the curve through the origin

$(0,0)$ is the level curve of $f$ when the

$K = 0$

2. $f(x,y) = (x-2)^2 + (y+3)^2$ $k = 1, 4$

Step: same like the first one.
IV. Find the equation of the level curve of \( f \) that contains the point \( P \).

\[
f(x,y)=y \arctan x \quad P(1,4)
\]

Solve:

Step 1: Put the P (1,4) into the function \( f \). \( 4 \arctan 1 = K \) then \( K = \pi \)

Step 2: Then we find the equation of the level curve of \( f \) is \( y \arctan x = \pi \)

V. Then we can find the equation of the level curve of \( f \) which with three variables.

\[
f(x,y,z)=x^2+4y^2-z^2 \quad P(2,-1,3)
\]

Step 1: \( f(2,-1,3)=2^2+4(-1)^2-3^2= -1 \)

Step 2: we find that the equation of the level curve of \( f \) is \( x^2+4y^2-z^2= -1 \)

VI. Describe the level surface of \( f \) for the given values of \( K \).

1. \( f(x,y,z)=x^2+y^2+z^2 \quad K= -1,0,4 \)

Solve:

For \( K= -1 \) \( f(x,y,z)=x^2+y^2+z^2= -1 \) is not right, because \( f(x,y,z)=x^2+y^2+z^2 \geq 0 \). So there is no level surface if \( K= -1 \).

For \( K= 0 \) \( f(x,y,z)=x^2+y^2+z^2=0 \). We get the solution that \( x=0, y=0, z=0 \). So the level surface is the origin \((0,0,0)\).
For K = 4 \( f(x,y,z) = x^2 + y^2 + z^2 = 4 \). We get the level surface is with center (0,0,0) and radium 2.

**Limits of functions with several variables**

Let the function \( f \) of two variables be defined throughout the interior of a circle with center (a,b), expect possibly at (a,b) itself. The statement \( \lim f(x,y) = L \) means that for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that \( (x,y) \to (a,b) \)

If \( 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \) then \( |f(x,y) - L| < \epsilon \) (SWOKOWSKI)

We can use the definition to solve some kinds of problems.

**TWO-PATH RULE:**

If two different paths to a point P (a,b) produce two different limiting values for \( f \), then \( \lim_{(x,y) \to (a,b)} f(x,y) \) does not exist.

I. Find the limit.

1. \( \lim_{(x,y) \to (0,0)} \frac{x^2 - 2}{3 + xy} \)

   Solve: We just need to put the approximate value of \( x,y \) into function \( \frac{x^2 - 2}{3 + xy} \).

   \( \frac{0^2 - 2}{3 + 0*0} = -2/3 \)

   So the result is \(-2/3\)

2. \( \lim_{(x,y) \to (0,0)} \frac{x^4 - y^4}{x^2 + y^2} \)

   Solve: We find that if we put \( x=0 \) and \( y=0 \) into the \( \frac{x^4 - y^4}{x^2 + y^2} \), the denominator will be 0, so we cannot do that. However, we find that there is a same factor we can delete it. \( \frac{x^4 - y^4}{x^2 + y^2} = \frac{(x^2 + y^2)(x^2 - y^2)}{(x^2 + y^2)} \)

   Then we can put the \( x=0 \) and \( y=0 \) into the \( (x^2 - y^2) \).

   \( 0^2 - 0^2 = 0 \)

   So the result is 0.
II. Show that the limit does not exist.

1. \( \lim_{(x,y) \to (0,0)} \frac{2x^2 - y^2}{x^2 + 2y^2} \)

Solve: We can use the two paths method. We need to find two different paths. If we get two different limits, it is showing that the limit does not exist.

Let \( y = 0 \) \( \lim_{(x,y) \to (0,0)} \frac{2x^2 - y^2}{x^2 + 2y^2} = 2 \)

Let \( y = x \) \( \lim_{(x,y) \to (0,0)} \frac{2x^2 - y^2}{x^2 + 2y^2} = 0 \)

\( 0 \neq 2 \) So the limit does not exist.

2. \( \lim_{(x,y) \to (0,0)} \frac{3x^2y^3}{2y^5 - 2x^5} \)

Solve:

Let \( y = 0 \) \( \lim_{(x,y) \to (0,0)} \frac{3x^2y^3}{2y^5 - 2x^5} = 0 \)

Let \( y = 2x \) \( \lim_{(x,y) \to (0,0)} \frac{3x^2y^3}{2y^5 - 2x^5} = 31 \)

\( 0 \neq 31 \) So the limit does not exist.

3. \( \lim_{(x,y,z) \to (0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2} \)

Solve:

Let \( x = y = 0 \) \( \lim_{(x,y,z) \to (0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2} = 0 \)

Let \( x = y = z \) \( \lim_{(x,y,z) \to (0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2} = 1 \)

\( 0 \neq 1 \) So the limit does not exist.

**Partial derivatives**

**First Partial Derivative**

we defined the derivative \( f'(x) \) of a function of one variable as \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \)
Let $f$ be a function of two variables. The first partial derivative of $f$ with respect to $x$ and $y$ are the function $f_x$ and $f_y$ such that

$$f_x(x,y) = \lim_{h \to 0} f(x + h, y) - f(x, y)/h$$  
$$f_y(x,y) = \lim_{h \to 0} f(x, y + h) - f(x, y)/h$$

Notation of Partial Derivatives

$$Z = f_x (x,y) = \partial f / \partial x = \partial z / \partial x = Z_x$$

$$Z = f_y (x,y) = \partial f / \partial y = \partial z / \partial y = Z_y$$

Now we discuss how to use the partial derivatives method to find the derivatives of function with several variables.

I. Find the first partial derivatives of $f$.

The method: we need to treat other variables as constants.

1. $f(x,y) = 2x^4y^3-xy^2+3y+1$

   solve:

   $$f_x (x,y) = 8x^4y^3-3y^2$$

   $$f_y (x,y) = 6x^4y^2-2xy+3$$

2. $f(r,s) = \sqrt{r^2 + s^2}$

   solve:

   Firstly, we can rewrite $f(r,s) = \sqrt{r^2 + s^2} = (r^2+s^2)^{1/2}$

   $$f_r (r,s) = (1/2)* (r^2+s^2)^{-1/2} r = r/(r^2+s^2)^{1/2}$$

   $$f_s (r,s) = (1/2)* (r^2+s^2)^{-1/2} s = s/(r^2+s^2)^{1/2}$$

3. $f(x,y) = xe^y + y\sin x$

   solve:

   $$f_x (x,y) = e^y + y\cos x$$

   $$f_y (x,y) = xe^y + \sin x$$
4. \( f(x, y, z) = 3x^2z + x^y^2 \)

solve:

\( f_x(x, y, z) = 6xz + y^2 \)

\( f_y(x, y, z) = 2xy \)

\( f_z(x, y, z) = 3x^y^2 \)

5. \( f(r, s, t) = r^2 e^t (2s) \cos t \)

\( f_r(r, s, t) = 2re^t (2s) \cos t \)

\( f_s(r, s, t) = 2r^2 e^t (2s) \cos t \)

\( f_t(r, s, t) = -r^2 e^t (2s) \sin t \)

**Second Partial Derivative**

If \( W = f(x, y) \), we write

\( (\partial^2 / \partial x^2) f(x, y) = f_{xx}(x, y) = \partial^2 W / \partial x^2 = W_{xx} \)

\( (\partial^2 / \partial y \partial x) f(x, y) = f_{xy}(x, y) = \partial^2 W / \partial y \partial x = W_{xy} \)

Now let me show some questions.

1. Verify that \( W_{xy} = W_{yx} \)

   1. \( W = x y^4 - 2 x^2 y^3 + 4 x y^2 - 3y \)

   \( W_x = f_x(x, y) = y^4 - 4 x y^3 + 8x \)

   \( W_y = f_y(x, y) = 4 x y^3 - 6 x^2 y^2 - 3 \)

   \( W_{xy} = (\partial / \partial y) f_x(x, y) = (\partial / \partial y) (y^4 - 4 x y^3 + 8x) = 4y^3 - 12xy^2 \)

   \( W_{yx} = (\partial / \partial x) f_y(x, y) = (\partial / \partial x) (4 x y^3 - 6 x^2 y^2 - 3) = 4y^3 - 12xy^2 \)

   So \( W_{xy} = W_{yx} \)

2. \( W = x^3 e^{(-2y)} + y^4 (-2) \cos x \)
\[ W_x = fx(x,y) = 3x^2 e^{(-2y)} - y^2 \sin x \]

\[ W_y = fy(x,y) = (-2)x^3 e^{(-2y)} + (-2)y^3 \cos x \]

\[ W_{xy} = \left( \frac{\partial}{\partial y} \right) fx(x,y) = \left( \frac{\partial}{\partial y} \right) (3x^2 e^{(-2y)} - y^2 \sin x) \]

\[ = (-6)x^2 e^{(-2y)} + 2y^3 \sin x \]

\[ W_{yx} = \left( \frac{\partial}{\partial x} \right) fy(x,y) = \left( \frac{\partial}{\partial x} \right) ((-2)x^3 e^{(-2y)} + (-2)y^3 \cos x) \]

\[ = (-6)x^2 e^{(-2y)} + 2y^3 \sin x \]

So \( W_{xy} = W_{yx} \)

II. If \( W = 3x^2 y^3 z + 2x y^4 z^2 - yz \), find \( W_{xyz} \).

Solve:

\[ W_x = fx(x,y,z) = 6x y^3 z + 2 y^4 z^2 \]

\[ W_{xy} = \left( \frac{\partial}{\partial y} \right) fx(x,y,z) \]

\[ = (6x y^3 z + 2 y^4 z^2) \]

\[ = 18x y^2 z + 8y^3 z^2 \]

\[ W_{xyz} = \left( \frac{\partial^2}{\partial z \partial y} \right) fx(x,y,z) = \left( \frac{\partial}{\partial z} \right) W_{xy} \]

\[ = (18x y^2 z + 8y^3 z^2) \]

\[ = 18x y^2 z + 16y^3 z \]

III. If \( w = \sin xy \), find \( \frac{\partial^3}{\partial z \partial y \partial x} w \).

\[ \frac{\partial^3}{\partial z \partial y \partial x} w \]

\[ (\frac{\partial}{\partial z}) f(x,y,z) = xy \cos xyz \]

\[ f_{xy}(x,y,z) = (\frac{\partial}{\partial y})(\frac{\partial}{\partial z}) f(x,y,z) = (\frac{\partial}{\partial y})(xy \cos xyz) \]

\[ = x \cos xyz - x^2 yz \sin xyz \]