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Center of Mass

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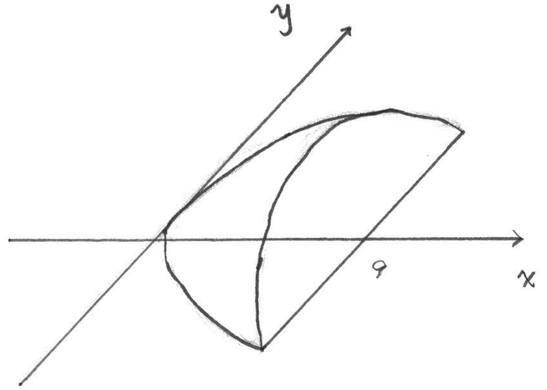
Hornors Project – Center of Mass

In Chapter 6, we have learnt how to find the center of mass when solid has a constant density (ρ). So, in this project, we will replace a constant density with a function: (1) $\rho(x)$, (2) $\rho(y)$, (3) $\rho(x, y)$.

For these problems, we need to use double integrals. So, what is double integral? In 6.4, we learnt the volume-by-cross-section formula:

$$V = \int_a^b A(x)\,dx = \lim_{\|P'\| \to 0} \sum_k A(\mu_k)\,\Delta x_k,$$

Ex: R: bounded by $x = y^2$ and $x = 9$. Solid: base R, every cross-section by a plane prependicular to thexaxis is a semicircle. Find the volume.

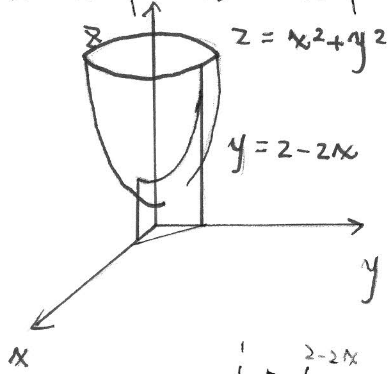


$$A(x) = \tfrac{1}{2}\pi (\sqrt{x})^2 = \tfrac{1}{2}\pi x$$

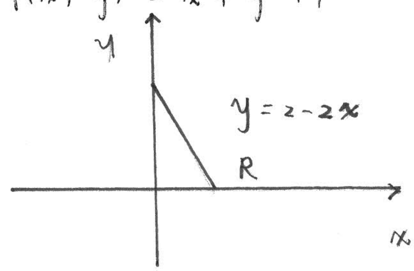
$$V = \tfrac{1}{2}\int_0^9 \pi x \, dx = \tfrac{1}{4}\pi x^2 \Big]_0^9 = \frac{81}{4}\pi$$

For this problem, we know how to calculate the area of semicircle. However, for some problems, we can not find a formula to calculate the area directly. So, we need to use integral to find the area. And then, use another integral to get the volume. That's why we need to use double integrals.

Ex: Find the volume V of the solid in the first octant bounded by the coordinate planes, the paraboloid $z = x^2+y^2+1$, and the plane $2x+y=2$



$f(x,y) = x^2+y^2+1$



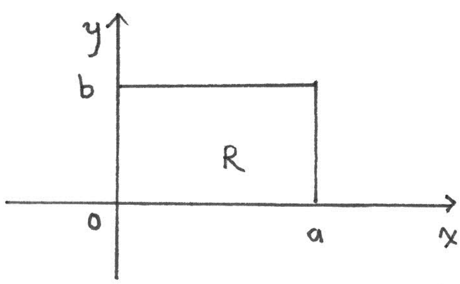
$$\therefore\ V = \int_0^1 \left[\int_0^{2-2x} (x^2+y^2+1)\, dy \right] dx = \int_0^1 \left[x^2 y + \frac{1}{3}y^3 + y \right]_0^{2-2x} dx$$

$$= \int_0^1 \left(1 - \frac{14}{3}x^3 + 10x^2 - 10x + \frac{14}{3}\right) dx$$

$$= \left[-\frac{7}{6}x^4 + \frac{10}{3}x^3 - 5x^2 + \frac{14}{3}x \right]_0^1 = \frac{11}{6}$$

Areas can also be considered as limits of double integrals.

Ex:



$f(x,y) = 1$

$$\iint_R f(x,y)\, dA = \int_0^b \int_0^a 1\, dx\, dy$$

$$= \int_0^b [x]_0^a\, dy = \int_0^b a\, dy = ay \Big]_0^b = ab = \text{Area}$$

Rectangle Area $= a \times b = ab$

Now, we know what is double integrals. And then, how can we use double integrals to calculate the center of mass. From 17.6 (P926), we can get the formulas about the moments and mass.

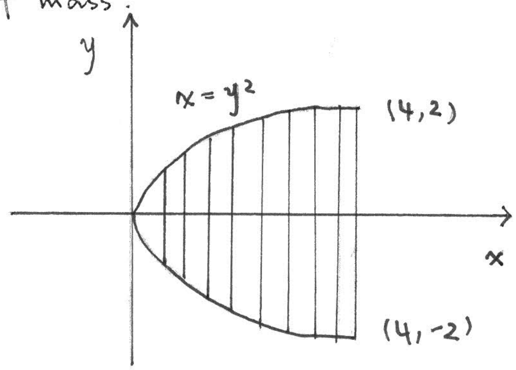
(i) $m = \iint_R \delta(x, y)\, dA$

(ii) $M_x = \iint_R y\,\delta(x, y)\, dA$, $\quad M_y = \iint_R x\,\delta(x, y)\, dA$

(iii) $\bar{x} = \dfrac{M_y}{m} = \dfrac{\iint_R x\,\delta(x,y)\,dA}{\iint_R y\,\delta(x,y)\,dA} \qquad \bar{y} = \dfrac{M_x}{m} = \dfrac{\iint_R y\,\delta(x,y)\,dA}{\iint_R \delta(x,y)\,dA}$

I. In x-direction, has different density; in y-direction has same density

Ex: A lamina has the shape of the region R in the xy-plane bounded by the parabola $x = y^2$ and the line $x = 4$. The area mass density at the Point $P(x, y)$ is directly proportional to the distance from the y-axis to P. Find the center of mass.



the area mass density at (x, y) is $\delta(x, y) = kx$

$\bar{y} = 0$

$m = \int_{-2}^{2}\int_{y^2}^{4} kx\,dx\,dy = k\int_{-2}^{2}\left[\frac{1}{2}x^2\right]_{y^2}^{4}$

$= \frac{1}{2}k\int_{-2}^{2}(16 - y^4)\,dy = \frac{128}{5}k$

$M_y = \iint_R x\,\delta(x, y)\,dA = \int_{-2}^{2}\int_{y^2}^{4} kx^2\,dx\,dy = k\int_{-2}^{2}\left[\frac{1}{3}x^3\right]_{y^2}^{4} dy$

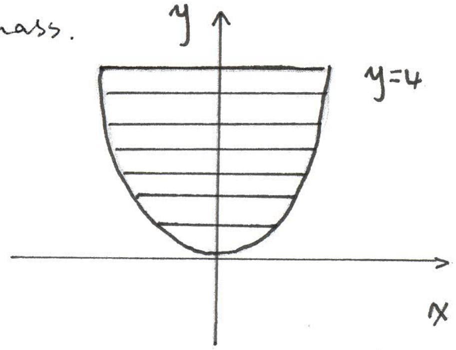
$= \frac{1}{3}k\int_{-2}^{2}(64 - y^6)\,dy = \frac{512}{7}k$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{512k}{7}\cdot\dfrac{5}{128k} = \dfrac{20}{7} \approx 2.86$

$\therefore$ the center of mass is $\left(\frac{20}{7}, 0\right)$

II. In x-direction, has the same density; in y-direction has different density.

Ex: A lamina has the shape of the region R in the xy - plane bounded by the parabola $y=x^2$ and the line $y=4$. The area mass density at the point $P(x,y)$ is directly proportional to the distance from the x-axis to P. Find the center of mass.



the area mass density at (x,y) is

$\delta(x,y) = ky \quad \bar{x}=0$

$$m = \iint_R \delta(x,y)\, dA = \int_{-2}^{2}\int_{x^2}^{4} ky\, dy\, dx = k\int_{-2}^{2} \tfrac{1}{2}y^2\Big]_{x^2}^{4} dx$$

$$= k\int_{-2}^{2} 8-\tfrac{1}{2}x^4\, dx = \left[-\tfrac{1}{10}x^5+8x\right]_{-2}^{2} = \frac{64}{5}\times 2 = \frac{128}{5}$$

$$M_x = \iint_R y\,\delta(x,y)\, dA = k\int_{-2}^{2}\int_{x^2}^{4} y^2\, dy\, dx$$

$$= \int_{-2}^{2}\int_{x^2}^{4} y^2\, dy\, dx = \int_{-2}^{2}\left(\tfrac{1}{3}y^3\right]_{x^2}^{4} dx = \int_{-2}^{2}\left(\frac{64}{3}-\frac{1}{3}x^6\right) dx$$

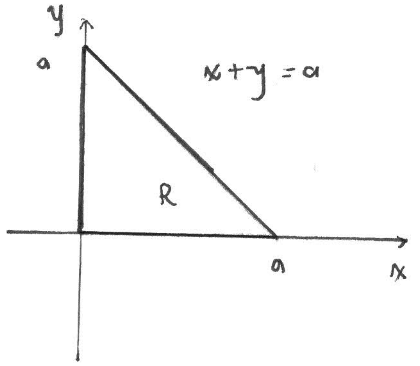
$$= \left[\frac{64}{3}x - \frac{1}{21}x^7\right]_{-2}^{2} = \frac{512}{7}$$

$$\bar{y} = \frac{M_x}{m} = \frac{512}{7}\times\frac{5}{128} = \frac{20}{7}$$

$\therefore$ the center of mass is $\left(0, \frac{20}{7}\right)$

III. In x-direction and y-direction, both of them have different density.

Ex: A lamina has the shape of an isosceles right triangle with equal sides of length a. The area mass density at a Point P is directly proportional to the square of the distance from P to the vertex that is opposite the hypotenuse. Find the center of mass.



Using the area mass density $\delta(x,y) = k(x^2+y^2)$

$$m = \iint_R \delta(x,y)\,dA = \int_0^a \int_0^{a-x} k(x^2+y^2)\,dy\,dx$$

$$= k\int_0^a \left[x^2 y + \tfrac{1}{3}y^3\right]_0^{a-x} dx = k\int_0^a \left[ax^2 - x^3 + \tfrac{1}{3}(a-x)^3\right] dx$$

$$= k\left[-\tfrac{1}{4}x^4 + \tfrac{1}{3}ax^3 - \tfrac{1}{12}(a-x)^4\right]_0^a = \tfrac{1}{6}ka^4$$

$$M_y = \iint_R x\,\delta(x,y)\,dA = \int_0^a\int_0^{a-x} xk(x^2+y^2)\,dy\,dx$$

$$= \int_0^a\int_0^{a-x} kx(x^2+y^2)\,dy\,dx = k\int_0^a\int_0^{a-x}[x^3 + xy^2]\,dy\,dx = k\int_0^a \left[x^3y + \tfrac{1}{3}xy^3\right]_0^{a-x}$$

$$= k\int_0^a \left[x^4 - ax^3 + \tfrac{1}{3}(x-a)^3\right]dx = k\left(\tfrac{1}{5}x^5 - \tfrac{1}{4}ax^4 + \tfrac{1}{15}x^5 - \tfrac{1}{4}ax^4 + \tfrac{1}{3}a^2x^3 - \tfrac{1}{6}a^3x^2\right]_0^a$$

$$= \tfrac{1}{15}ka^5$$

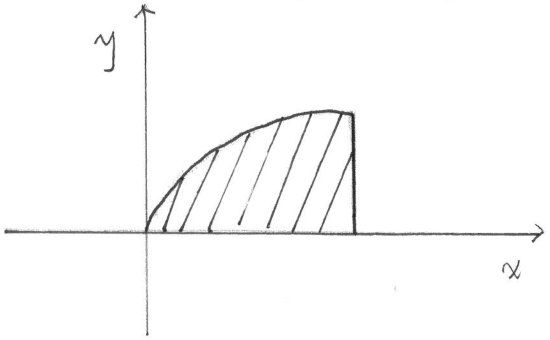
$$\therefore\ \bar{x} = \frac{\tfrac{1}{15}ka^5}{\tfrac{1}{6}ka^4} = \tfrac{2}{5}a$$

Similarly, $M_x = \tfrac{1}{15}ka^5$ and $\bar{y} = \tfrac{2}{5}a$.

Thus, the center of mass of the lamina is $\left(\tfrac{2}{5}a, \tfrac{2}{5}a\right)$

Exercise: Find the mass and the center of mass of the lamina that has the shape of the region bounded by the graph of the given equations and has the indicated area mass density.

11) $y = \sqrt{x}$, $x = 9$, $y = 0$; $\delta(x, y) = x + y$



$$m = \iint_R \delta(x, y)\, dx\, dy$$

$$= \int_0^9 \int_0^{\sqrt{x}} (x+y)\, dy\, dx = \int_0^9 (x^{\frac{3}{2}} + \tfrac{1}{2}x]\, dx$$

$$= [\tfrac{2}{5}x^{\frac{5}{2}} + \tfrac{1}{4}x^2]_0^9 = \frac{2349}{20}$$

$$M_y = \iint_R x\,\delta(x, y)\, dA = \int_0^9 \int_0^{\sqrt{x}} x(x+y)\, dy\, dx = \int_0^9 (x^2 y + \tfrac{1}{2}xy^2]_0^{\sqrt{x}}\, dx$$

$$= \int_0^9 (x^{\frac{5}{2}} + \tfrac{1}{2}x^2)\, dx = [\tfrac{2}{7}x^{\frac{7}{2}} + \tfrac{1}{6}x^3]_0^9 = \frac{31347}{42}$$

$$M_x = \iint_R y\,\delta(x, y)\, dA = \int_0^9 \int_0^{\sqrt{x}} y(x+y)\, dy\, dx = \int_0^9 (\tfrac{1}{2}xy^2 + \tfrac{1}{3}y^3)_0^{\sqrt{x}}\, dx$$

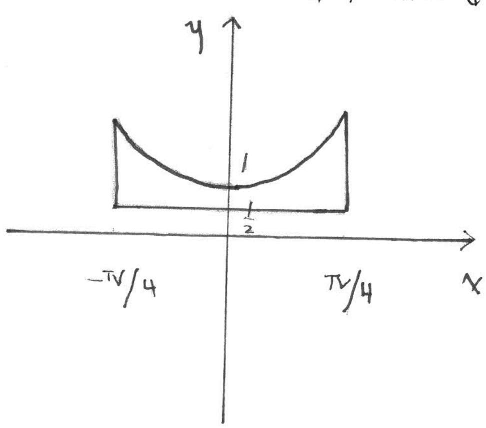
$$= \int_0^9 (\tfrac{1}{2}x^2 + \tfrac{1}{3}x^{\frac{3}{2}})\, dx = [\tfrac{1}{6}x^3 + \tfrac{2}{15}x^{\frac{5}{2}}]_0^9 = \frac{4617}{30}$$

$$\bar{x} = \frac{M_y}{m} = \frac{31347}{42} \times \frac{20}{2349} = \frac{1290}{203}$$

$$\bar{y} = \frac{M_x}{m} = \frac{4617}{30} \times \frac{20}{2349} = \frac{38}{29}$$

$\therefore$ the center of mass is $\left(\frac{1290}{203}, \frac{38}{29}\right)$

12) $y = \sec x$, $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$, $y = \frac{1}{2}$; $\delta(x, y) = 4$



$$\bar{x} = 0$$

$$m = \iint_R \delta(x, y)\, dA$$

$$= \int_{-\pi/4}^{\pi/4} \int_{\frac{1}{2}}^{\sec x} 4\, dy\, dx = \int_{-\pi/4}^{\pi/4} (4\sec x - 2)\, dx$$

$$= 4\ln(\sqrt{2}+1) - 4\ln(\sqrt{2}-1) - \pi \approx 3.91$$

$$M_x = \iint_R y\delta(x, y) = \int_{-\pi/4}^{\pi/4} \int_{\frac{1}{2}}^{\sec x} 4y \, dy \, dx = \int_{-\pi/4}^{\pi/4} \int_{\frac{1}{2}}^{\sec x} 4y \, dy \, dx$$

$$= \int_{-\pi/4}^{\pi/4} 2y^2 \, dy \, dx = \int_{-\pi/4}^{\pi/4} (2\sec^2 x - \tfrac{1}{2}) \, dx = \left[2\tan(x) - \tfrac{1}{2}x \right]_{-\pi/4}^{\pi/4}$$

$$= (2 - \tfrac{\pi}{8}) \times 2 = 4 - \tfrac{\pi}{4}$$

$\bar{y} = \frac{M_x}{m} \approx 0.822$ $\therefore$ the center of mass is $(0, 0.822)$